

REAL AFFINE VARIETIES OF NONNEGATIVE CURVATURE

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ABSTRACT. Let $X_{\mathbb{C}}$ be a smooth real affine variety with compact real points $X_{\mathbb{R}}$. We show that $X_{\mathbb{C}}$ is diffeomorphic to the normal bundle of $X_{\mathbb{R}}$ provided that $X_{\mathbb{C}}$ admits a complete Riemannian metric of nonnegative sectional curvature which is also invariant under the conjugation.

1. INTRODUCTION

In [7], Totaro asked whether a compact manifold M with nonnegative sectional curvature has a good complexification, i.e. there exists a smooth real affine variety $X_{\mathbb{C}}$ such that M is diffeomorphic to $X_{\mathbb{R}}$ (the real points of $X_{\mathbb{C}}$) and the inclusion $X_{\mathbb{R}} \rightarrow X_{\mathbb{C}}$ is a homotopy equivalence. A positive answer to Totaro's question would in particular resolves an odd problem of Hopf saying that a compact nonnegatively curved manifold has nonnegative Euler number [3]. Recently I. Biswas and M. Mj has obtained a positive answer to Totaro's question for three dimensional manifolds [1].

According to a remarkable theorem of Nash and Tognoli [5] [6], any compact differentiable manifold can be realized as the real points $X_{\mathbb{R}}$ of a smooth real affine variety $X_{\mathbb{C}}$. Now Totaro's question can be reformatted as the following:

Let $X_{\mathbb{C}}$ be a smooth real affine variety with compact real points $X_{\mathbb{R}}$. If $X_{\mathbb{R}}$ admits a complete Riemannian metric of nonnegative sectional curvature, does there exist a smooth real affine variety $U_{\mathbb{C}}$ such that $X_{\mathbb{R}}$ is diffeomorphic to $U_{\mathbb{R}}$ and the inclusion $U_{\mathbb{R}} \rightarrow U_{\mathbb{C}}$ is a homotopy equivalence?

In this paper we give a positive answer to the above question under a stronger assumption. More precisely, we are going to prove the following

Theorem 1.1. *Let $X_{\mathbb{C}}$ be a smooth real affine variety with compact real points $X_{\mathbb{R}}$. If $X_{\mathbb{C}}$ admits a complete Riemannian metric g of nonnegative sectional curvature such that $\tau^*g = g$, where $\tau : X_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$ is the conjugation, then $X_{\mathbb{C}}$ is diffeomorphic to the normal bundle of $X_{\mathbb{R}}$. In particular, the inclusion $X_{\mathbb{R}} \rightarrow X_{\mathbb{C}}$ is a homotopy equivalence.*

Note that in Theorem 1.1, $X_{\mathbb{R}}$ is the fixed point set of the isometry τ . It follows that $X_{\mathbb{R}}$ is a totally geodesic submanifold of $X_{\mathbb{C}}$, in particular, the induced metric of g has nonnegative sectional curvature on $X_{\mathbb{R}}$.

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Remark 1.2. The assumption that $X_{\mathbb{R}}$ is compact in Theorem 1.1 can not be dropped. An example is given by $X_{\mathbb{C}} = \{(z_1, z_2) \in \mathbb{C}^2 | z_1^2 - z_2^2 = 1\}$. Define a map F by

$$F : X_{\mathbb{C}} \rightarrow \mathbf{S}^1 \times \mathbb{R}, (z_1, z_2) \mapsto \left(\frac{z_1 + z_2}{|z_1 + z_2|}, \ln|z_1 - z_2| \right).$$

Then F is a diffeomorphism. By pulling back the product metric on $\mathbf{S}^1 \times \mathbb{R}$, we see that $X_{\mathbb{C}}$ admits a complete flat Riemannian metric which is also invariant under the conjugation. However, the inclusion $X_{\mathbb{R}} \rightarrow X_{\mathbb{C}}$ is *not* a homotopy equivalence since $X_{\mathbb{C}}$ is connected but $X_{\mathbb{R}}$ has two noncompact connected components.

Example 1.3. Let G be a closed subgroup of $U(n)$ and $G_{\mathbb{C}}$ be the complexification of G . It's well known that $G_{\mathbb{C}}$ is a real affine variety [3]. Note that there is a diffeomorphism $F : G \times \text{Lie}(G) \rightarrow G_{\mathbb{C}}$ given by

$$F : G \times \text{Lie}(G) \rightarrow G_{\mathbb{C}}, (x, A) \mapsto e^{iA}x,$$

where $\text{Lie}(G)$ is the Lie algebra of G .

Let $\tau : G_{\mathbb{C}} \rightarrow G_{\mathbb{C}}$ be the conjugation and $\phi : G \times \text{Lie}(G) \rightarrow G \times \text{Lie}(G), (x, A) \mapsto (\bar{x}, -\bar{A})$ be an involution. Then it is direct to check that $\tau F = F\phi$. Let h be a bi-invariant metric on G which is also invariant under the conjugation. By pulling back the product of h and the flat metric on $\text{Lie}(G)$, we get a complete Riemannian metric g on $G_{\mathbb{C}}$ with nonnegative sectional curvature such that $\tau^*g = g$. However, in general g is *not* $G_{\mathbb{C}}$ invariant as F is *not* a group homomorphism in general.

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2. PROOF OF THEOREM 1.1

The proof of Theorem 1.1 is based on the soul construction of Cheeger and Gromoll in [2]. In fact, we are going to show that $X_{\mathbb{R}}$ is a soul of $X_{\mathbb{C}}$.

We first prove the following theorem which is interesting on its own.

Theorem 2.1. *Let (E, g) be a complete noncompact connected manifold with non-negative sectional curvature and f be an isometry of (E, g) with E^f compact, where E^f is the fixed point set of f . Then E^f is contained in a soul of E .*

Proof. Fix $x_0 \in E^f$ and define $B(x) = \sup_{\gamma} B_{\gamma}(x), x \in E$, where $\gamma : [0, +\infty) \rightarrow E$ is a ray with $\gamma(0) = x_0$ and B_{γ} is the Busemann function given by

$$B_{\gamma}(x) = \lim_{t \rightarrow +\infty} (t - d(x, \gamma(t))).$$

Since f is an isometry with $f(x_0) = x_0$, we see $B_{f \circ \gamma}(f(x)) = B_{\gamma}(x)$. Then we get $B \circ f = B$ which will be crucial for us. By the arguments in [2], we get a soul S of E which satisfies $f(S) = S$ (e.g, Corollary 6.3 in [2]). Moreover, there is a distance nonincreasing retraction $P : E \rightarrow S$ (e.g, [8]). By checking the construction of P carefully and using $B \circ f = B$, we also see

$$P \circ f = f \circ P.$$

For any $x \in E^f$, we have $f(x) = x$ and

$$d(P(x), f \circ P(x)) = d(P(x), P \circ f(x)) \leq d(x, f(x)) = 0.$$

Then $P(x) = f \circ P(x)$ and hence

$$(2.1) \quad P(E^f) \subseteq E^f.$$

It follows that $P(E^f) \subseteq E^f \cap S$. On the other hand, since $P|_S = Id|_S$, we see $E^f \cap S = P(E^f \cap S) \subseteq P(E^f)$. Then

$$P(E^f) = E^f \cap S.$$

Let $H : E \times [0, 1] \rightarrow E$ be the homotopy between $Id|_E$ and P constructed in [8]. Then for each $s \in [0, 1]$, $H_s := H(\cdot, s) : E \rightarrow E$ is a distance nonincreasing map. By checking the construction of H carefully and using $B \circ f = B$, we see

$$H_s \circ f = f \circ H_s.$$

By a similar argument in the proof of 2.1, we see $H_s(E^f) \subseteq E^f$ and hence $H|_{E^f \times [0, 1]} : E^f \times [0, 1] \rightarrow E^f$ is a homotopy between $Id|_{E^f}$ and $P|_{E^f}$. It follows that $E^f \cap S$ is a retraction of E^f . Since E^f is a compact manifold without boundary, we see $E^f \cap S = E^f$ and hence $E^f \subseteq S$. \square

Now Theorem 1.1 follows from Theorem 2.1 easily. Without loss of generality, we can assume that $X_{\mathbb{C}}$ is connected. Then by Theorem 2.1, $X_{\mathbb{R}}$ is contained in a soul S of $X_{\mathbb{C}}$. It's known that a smooth real affine variety $X_{\mathbb{C}}$ has the homotopy type of a CW complex of dimension $\leq \dim(X_{\mathbb{R}})$ (Theorem 7.2 in [4]). It follows that $X_{\mathbb{R}}$ has the same dimension as S and then $X_{\mathbb{R}} = S$ since $X_{\mathbb{R}}$ is a closed submanifold of S and S is connected.

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